MATHEMATISCHE NACHRICHTEN

www.mn-journal.org

Founded in 1948 by Erhard Schmidt

WILEY

REPRINT

Math. Nachr. **288,** No. 5–6, 593–603 (2015) / **DOI** 10.1002/mana.201400210

Variational formula related to the self-affine Sierpinski carpets

Yongxin Gui[*1], **Wenxia Li**[**2], and **Dongmei Xiao**[***3]

[1] School of Mathematics and Statistics, HuBei University of Science and Technology, Xianning 437100, P. R. China

[2] Department of Mathematics, Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, P. R. China

[3] Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, P. R. China

Received 18 June 2014, accepted 19 June 2014
Published online 16 October 2014

Key words Self-affine Sierpinski carpet, digit frequency, group frequency, Hausdorff dimension
MSC (2010) Primary: 28A80, Secondary: 28A78

We consider those subsets of the self-affine Sierpinski carpets that are the union of an uncountable number of sets each of which consists of the points with their location codes having prescribed group frequencies. It is proved that their Hausdorff dimensions equal to the supremum of the Hausdorff dimensions of the sets in the union. The main advantage is that we treat these subsets in a unified manner and the value of the Hausdorff dimensions do not need to be guessed a priori.

1 Introduction and statement of main results

Let T be the expanding endomorphism of the 2-torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ given by the diagonal matrix $\mathrm{diag}(n, m)$ where $2 \le m \le n$ are integers. The simplest invariant sets for T, called the *self-affine Sierpinski carpets*, have the form

$$K(T, D) = \left\{ \sum_{k=1}^{\infty} \mathrm{diag}(n^{-k}, m^{-k}) d_k : \ d_k \in D \text{ for all } k \ge 1 \right\},$$

where $D \subseteq I \times J$ is the set of digits with $I = \{0, 1, \dots, n-1\}$ and $J = \{0, 1, \dots, m-1\}$. Alternatively, define a map $K_T : D^{\mathbf{N}} \to \mathbf{T}^2$ by

$$K_T((d_k)_{k=1}^{\infty}) = \sum_{k=1}^{\infty} \mathrm{diag}\big(n^{-k}, m^{-k}\big) d_k.$$

Then $K(T, D) = K_T(D^{\mathbf{N}})$. The set $K(T, D)$ were first studied by McMullen [13] and Bedford [2] independently. In the past two decades, some further problems related to the self-affine Sierpinski carpet $K(T, D)$ and its various variations have been proposed and considered by a large number of authors (see [1], [3]–[7], [9]–[12], [14]–[17] etc.)

For any $x = (x_i)_{i=1}^{\infty} \in D^{\mathbf{N}}$ and a nonempty subset $\Gamma \subseteq D$, define

$$N_k(x, \Gamma) = |\{1 \le j \le k : x_j \in \Gamma\}|, \tag{1.1}$$

where and throughout this paper we use $|A|$ to denote the number of members of a finite set A. Whenever there exists the limit

$$f(x, \Gamma) := \lim_{k \to \infty} \frac{N_k(x, \Gamma)}{k} \tag{1.2}$$

* e-mail: admireyou@163.com, Phone: +86 71 5833 8950, Fax: +86 71 5833 8007
** Corresponding author: e-mail: wxli@math.ecnu.edu.cn, Phone: +86 21 5268 2621, Fax: +86 21 5268 2621
*** e-mail: xiaodm@mail.sjtu.edu.cn, Phone: +86 21 5474 3147, Fax: +86 21 5474 3151

it is called the *group frequency* of Γ in the coding x. When we write the symbol $f(x,\Gamma)$ we are already assuming the existence of the limit in (1.2). When $\Gamma=\{d\}$ is a singleton, the symbols $N_k(x,\Gamma)$ and $f(x,\Gamma)$ are simplified as $N_k(x,d)$ and $f(x,d)$, respectively. In particular, $f(x,d)$ is called the *digit frequency* of d in the coding x. Clearly, if $f(x,d)$ exists for all $d\in\Gamma$ then $f(x,\Gamma)$ exists and equals to $\sum_{d\in\Gamma} f(x,d)$. But the converse is not true.

Let σ denote the projection of $\mathbf{R}^2$ onto its second coordinate. Let

$$B=\sigma(D) \text{ and } \alpha=\log_n m.$$

Let H be the set of probability vectors on D, i.e.,

$$H=\left\{\mathbf{p}=(p_d)_{d\in D} : 0\le p_d\le 1 \text{ and } \sum_{d\in D} p_d=1\right\}.$$

For a given probability vector $\mathbf{p}=(p_d)_{d\in D}\in H$, it induces a probability vector on B

$$\mathbf{q}=(q_b)_{b\in B} \quad\text{where}\quad q_b=\sum_{d\in D\cap(I\times\{b\})} p_d. \tag{1.3}$$

A probability vector $\mathbf{p}=(p_d)_{d\in D}\in H$ is called *uniformly distributed* on D if $p_d=1/|D|$. For $b\in B$, the set $D\cap(I\times\{b\})$ is called a *horizontal fiber* of D. D is said to have *uniform horizontal fibers* if $|D\cap(I\times\{b\})|$ is invariant for all $b\in B$.

Define a function on H

$$g(\mathbf{p})=-\alpha\sum_{d\in D} p_d\log_m p_d-(1-\alpha)\sum_{b\in B} q_b\log_m q_b, \tag{1.4}$$

where we adopt the convention that $0\log 0=0$. For a nonempty subset $A\subseteq H$ denote

$$g_{\sup}(A):=\sup_{\mathbf{p}\in A} g(\mathbf{p}) \quad\text{and}\quad g_{\max}(A):=\max_{\mathbf{p}\in A} g(\mathbf{p}) \quad\text{if the maximum is attainable.}$$

For a nonempty subset $\Sigma\subseteq H$ let

$$\Omega(\Sigma)=\left\{x=(x_i)_{i=1}^{\infty}\in D^{\mathbf{N}} : (f(x,d))_{d\in D}\in\Sigma\right\}, \tag{1.5}$$

i.e., $\Omega(\Sigma)$ consists of those sequences $x\in D^{\mathbf{N}}$ for which all digit frequencies $f(x,d), d\in D$, exist and $(f(x,d))_{d\in D}\in\Sigma$. When Σ is a singleton, Nielsen [14] proved

Theorem A. *Let $g(\mathbf{p})$ be defined as in* (1.4). *For $\mathbf{p}\in H$*

[R1] $\dim_H K_T(\Omega(\{\mathbf{p}\}))=\dim_P K_T(\Omega(\{\mathbf{p}\}))=g(\mathbf{p})$;

[R2] $\dim_B K_T(\Omega(\{\mathbf{p}\}))=\dim_B K_T(D^{\mathbf{N}})=\log_m\left(|B|^{1-\alpha}|D|^{\alpha}\right)$;

[R3] *Let γ denote the common value of* $\dim_H K_T(\Omega(\{\mathbf{p}\}))$ *and* $\dim_P K_T(\Omega(\{\mathbf{p}\}))$.

(a) *If $\mathbf{p}$ is uniformly distributed on D and if D has uniform horizontal fibers then*

$$0<\mathcal{H}^{\gamma}(K_T(\Omega(\{\mathbf{p}\})))\le\mathcal{P}^{\gamma}(K_T(\Omega(\{\mathbf{p}\})))<\infty;$$

(b) *If $\mathbf{p}$ is not uniformly distributed on D or if D does not have uniform horizontal fibers then*

$$\mathcal{H}^{\gamma}(K_T(\Omega(\{\mathbf{p}\})))=\mathcal{P}^{\gamma}(K_T(\Omega(\{\mathbf{p}\})))=\infty.$$

By (1.5) we have $K_T(\Omega(\Sigma))=\bigcup_{\mathbf{p}\in\Sigma} K_T(\Omega(\{\mathbf{p}\}))$. Our first goal is to prove the variational formula for the dimensions of $K_T(\Omega(\Sigma))$ (also see Theorem 3.1 in Section 3):

Theorem 1.1 *Let Σ be a nonempty subset of H. Then*

$$\dim_H K_T(\Omega(\Sigma))=\dim_P K_T(\Omega(\Sigma))=\sup_{\mathbf{p}\in\Sigma}\dim_H K_T(\Omega(\{\mathbf{p}\}))=g_{\sup}(\Sigma).$$

We also discuss the Hausdorff and packing measures of $K_T(\Omega(\Sigma))$ in their dimensions. By the continuity of $g(\mathbf{p})$, we know that the value $\sup_{\mathbf{p}\in\Sigma} g(\mathbf{p})$ can be attained on $\overline{\Sigma}$ (the closure of Σ). Let

$$\Sigma_{\sup}=\left\{\mathbf{p}^*\in\overline{\Sigma} : g(\mathbf{p}^*)=g_{\sup}(\Sigma)\right\}. \tag{1.6}$$

Then we have (also see Theorem 3.2 in Section 3):

Theorem 1.2 *Denote* $\gamma = \dim_H K_T(\Omega(\Sigma)) = \dim_P K_T(\Omega(\Sigma))$. *Let* $\Sigma_{\sup}$ *be defined as in* (1.6).

(I) *Suppose that* $\Sigma_{\sup} \cap \Sigma \neq \emptyset$. *We have*

(Ia) *If* $(1/|D|, \ldots, 1/|D|) \in \Sigma_{\sup} \cap \Sigma$ *and if D has uniform horizontal fibers, then* $0 < \mathcal{H}^\gamma(K_T(\Omega(\Sigma))) \leq \mathcal{P}^\gamma(K_T(\Omega(\Sigma))) < \infty$;

(Ib) *If* $(1/|D|, \ldots, 1/|D|) \in \Sigma_{\sup} \cap \Sigma$ *but D has not uniform horizontal fibers, then* $\mathcal{H}^\gamma(K_T(\Omega(\Sigma))) = \mathcal{P}^\gamma(K_T(\Omega(\Sigma))) = \infty$;

(Ic) *If* $(1/|D|, \ldots, 1/|D|) \notin \Sigma_{\sup} \cap \Sigma$, *then* $\mathcal{H}^\gamma(K_T(\Omega(\Sigma))) = \mathcal{P}^\gamma(K_T(\Omega(\Sigma))) = \infty$;

(II) *If* $\Sigma_{\sup} \cap \Sigma = \emptyset$, *then* $\mathcal{H}^\gamma(K_T(\Omega(\Sigma))) = \mathcal{P}^\gamma(K_T(\Omega(\Sigma))) = 0$.

Our next goal is to prove the variational formula for the case related to the group frequencies.

Let $\{\Gamma_i\}_{i=1}^s$ be a partition of D, i.e., all Γ_i are nonempty, pairwise disjoint and $D = \bigcup_{i=1}^s \Gamma_i$. For a nonempty subset Σ of

$$H_s = \left\{ \mathbf{c} = (c_i)_{i=1}^s : 0 \leq c_i \leq 1 \text{ and } \sum_{i=1}^s c_i = 1 \right\},$$

let

$$\Sigma^* = \left\{ \mathbf{p} = (p_d)_{d\in D} \in H : \left(\sum_{d\in\Gamma_i} p_d \right)_{i=1}^s \in \Sigma \right\} \tag{1.7}$$

and

$$\Omega_s(\Sigma) = \left\{ x = (x_i)_{i=1}^\infty \in D^{\mathbf{N}} : (f(x, \Gamma_i))_{i=1}^s \in \Sigma \right\}. \tag{1.8}$$

For $\mathbf{c} = (c_i)_{i=1}^s \in H_s$ one can check that the function $g(\mathbf{p})$ can attain its maximum on $\{\mathbf{c}\}^*$ (defined by (1.7) for $\Sigma = \{\mathbf{c}\}$) at a unique point $\mathbf{p} = (p_d)_{d\in D}$ satisfying

$$p_d = \frac{q_{\sigma(d)}^\theta}{\sum_{d'\in\Gamma_j} q_{\sigma(d')}^\theta} c_j, \quad \text{for } d \in \Gamma_j \text{ and } j = 1, 2, \ldots, s, \tag{1.9}$$

where $\theta = \frac{\alpha-1}{\alpha}$. For a vector $y = (y_i)_{i=1}^k \in \mathbf{R}^k$, $y > 0$ means that $y_i > 0$ for all $1 \leq i \leq k$ throughout this paper. When Σ is a singleton, Gui and Li [8] proved

Theorem B. *Let* $0 < \mathbf{c} = (c_i)_{i=1}^s \in H_s$. *Let* $\mathbf{p} = (p_d)_{d\in D}$ *be determined by* (1.9) *and let* $(q_b)_{b\in B}$ *be the probability vector induced by* $(p_d)_{d\in D}$ via (1.3). *Then*

$$\dim_H K_T(\Omega_s(\{\mathbf{c}\})) = g_{\max}(\{\mathbf{c}\}^*) = g(\mathbf{p}) = \alpha \sum_{j=1}^s \left(c_j \log_m \sum_{d\in\Gamma_j} q_{\sigma(d)}^\theta - c_j \log_m c_j \right),$$

where $\theta = \frac{\alpha-1}{\alpha} = 1 - \log_m n$.

When $s = |D|$, Theorem B reduces to [R1] of Theorem A. However, unlike [R1] the Hausdorff dimension of $K_T(\Omega_s(\{\mathbf{c}\}))$ $(s \neq |D|)$ generally doesn't coincide with its packing dimension. For example, when $s = 1$ (then $\mathbf{c} = (1)$) we have $\Omega_1(\{\mathbf{c}\}) = D^{\mathbf{N}}$ and $\dim_H K_T(\Omega_1(\{\mathbf{c}\})) < \dim_B K_T(\Omega_1(\{\mathbf{c}\})) = \dim_P K_T(\Omega_1(\{\mathbf{c}\}))$ except for some special cases. In fact, to our knowledge the packing dimension of $K_T(\Omega_s(\{\mathbf{c}\}))$ $(s \neq |D|)$ is unknown.

Now let us return to $K_T(\Omega_s(\Sigma))$. Obviously $K_T(\Omega_s(\Sigma)) = \bigcup_{\mathbf{c}\in\Sigma} K_T(\Omega_s(\{\mathbf{c}\}))$. We show that the variational formula holds for $\dim_H K_T(\Omega_s(\Sigma))$ (also see Theorem 3.3 in Section 3):

Theorem 1.3 *Let* Σ *be a nonempty subset of* H_s. *Let* Σ^* *and* $\Omega_s(\Sigma)$ *be defined as in* (1.7) *and* (1.8), *respectively. Then*

$$\dim_H K_T(\Omega_s(\Sigma)) = \sup_{\mathbf{c}\in\Sigma} \dim_H K_T(\Omega_s(\{\mathbf{c}\})) = g_{\sup}(\Sigma^*).$$

When $s = |D|$, Theorem 1.3 reduces to Theorem 1.1. Similar to Theorem 1.2, for $\emptyset \neq \Sigma \subseteq H_s$ let

$$\Sigma_{\sup} = \left\{ \mathbf{p} \in \overline{\Sigma}^* : g(\mathbf{p}) = g_{\sup}(\Sigma^*) \right\}. \tag{1.10}$$

By the same argument as in the proof of Theorem 1.2 (Theorem 3.2) we have

Theorem 1.4 *Denote $\gamma = \dim_H K_T(\Omega_s(\Sigma))$. Let $\Sigma_{\sup}$ be defined as in* (1.10).

(I) *Suppose that $\Sigma_{\sup} \cap \Sigma^* \neq \emptyset$. We have*
(Ia) *If $(1/|D|, \ldots, 1/|D|) \in \Sigma_{\sup} \cap \Sigma^*$ and if D has uniform horizontal fibers, then $0 < \mathcal{H}^\gamma(K_T(\Omega_s(\Sigma))) < \infty$;*
(Ib) *If $(1/|D|, \ldots, 1/|D|) \in \Sigma_{\sup} \cap \Sigma^*$ but D has not uniform horizontal fibers, then $\mathcal{H}^\gamma(K_T(\Omega_s(\Sigma))) = \infty$;*
(Ic) *If $(1/|D|, \ldots, 1/|D|) \notin \Sigma_{\sup} \cap \Sigma^*$, then $\mathcal{H}^\gamma(K_T(\Omega_s(\Sigma))) = \infty$;*
(II) *If $\Sigma_{\sup} \cap \Sigma^* = \emptyset$, then $\mathcal{H}^\gamma(K_T(\Omega_s(\Sigma))) = 0$.*

Below we list two examples as application of our theorems. In the most cases as in Example 1, the Hausdorff dimensions are implicitly determined. However, in some cases as in Example 2 the Hausdorff dimensions can be explicitly determined.

Example 1.5 Let A is a nonempty proper subset of D. Let

$$M = \left\{ x = (x_i)_{i=1}^\infty \in D^{\mathbf{N}} : 0 < \sum_{d \in A} f^2(x, d) \leq 2/3 \right\}.$$

Then by Theorem 1.3 we have

$$\dim_H K_T(M) = g_{\sup}(Q) = \sup_{(p_d)_{d\in D} \in Q} -\alpha \sum_{d \in D} p_d \log_m p_d - (1-\alpha) \sum_{b \in B} q_b \log_m q_b$$

where $Q = \left\{ \mathbf{p} = (p_d)_{d \in D} \in H : 0 < \sum_{d \in A} p_d^2 \leq 2/3 \right\}$.

Example 1.6 (See [6].). For two distinct horizontal fibres F_{b_1}, F_{b_2} and $\beta > 0$, let

$$M = \left\{ x = (x_i)_{i=1}^\infty \in D^{\mathbf{N}} : f(x, F_{b_1}) = \beta f(x, F_{b_2}) \right\}.$$

Let $Q = \left\{ \mathbf{p} = (p_d)_{d \in D} \in H : \sum_{d \in F_{b_1}} p_d = \beta \sum_{d \in F_{b_2}} p_d \right\}$. Then by Theorem 1.3 we have $\big($recall that $\alpha = \log_n m\big)$

$$\dim_H K_T(M) = g_{\sup}(Q) = \log_m \left((1+\beta) \left(\beta^{-\beta} |F_{b_1}|^{\alpha\beta} |F_{b_2}|^{\alpha} \right)^{\frac{1}{1+\beta}} + \sum_{b \in B \setminus \{b_1, b_2\}} |F_b|^\alpha \right).$$

The rest of this paper is arranged as follows. Some known results needed in the present paper and lemmas are given in the next section. We shall show that Theorem B is also true when $\mathbf{c} \geq 0$. The proofs of Theorems 1.1–1.2 and 1.3 are given in Section 3.

2 Preliminaries

For $x = (x_n)_{n=1}^\infty \in (I \times J)^{\mathbf{N}}$ and $k \in \mathbf{N}$, let $Q_k(x)$ consist of all points $K_T(y)$ where $y \in (I \times J)^{\mathbf{N}}$ are such that $y_j = x_j$ for $1 \leq j \leq [\alpha k]$ and $\sigma(y_j) = \sigma(x_j)$ for $[\alpha k] + 1 \leq j \leq k$ where $[t]$ denotes the greatest integer not more than t ($\in \mathbf{R}$). The sets $Q_k(x)$ are approximate squares in $[0,1]^2$, whose sides have length $n^{-[\alpha k]}$ and m^{-k}. Note that the radio of the sides of $Q_k(x)$ is at most n, and their diameters $\operatorname{diam} Q_k(x)$ satisfy

$$\sqrt{2} m^{-k} \leq \operatorname{diam} Q_k(x) \leq \sqrt{2} n m^{-k}.$$

As in [13], [14] [16], [17] one can use these approximate squares to calculate dimension since one can restrict attention to covers by such approximate squares.

The following two lemmas will be used in this paper, which are just reformation of Rogers-Taylor density theorem.

Lemma 2.1 ([14, Lemma 4]) *Suppose that μ is a finite Borel measure on $[0,1]^2$, and that E is a subset of $(I\times J)^{\mathbf{N}}$ such that $K_T(E)$ is a Borel subset of $[0,1]^2$, and $\mu(K_T(E))>0$. Let δ be a positive number. For each point $x\in E$, put*

$$A(x)=\limsup_{k\to\infty}(k\delta+\log_m\mu(Q_k(x))).$$

1) *If $A(x)=-\infty$ for all $x\in E$, then $\mathcal{H}^\delta(K_T(E))=+\infty$;*
2) *If $A(x)=+\infty$ for all $x\in E$, then $\mathcal{H}^\delta(K_T(E))=0$;*
3) *If there are real numbers a and b such that $a\le A(x)\le b$ for all $x\in E$, then $0<\mathcal{H}^\delta(K_T(E))<+\infty$.*

Lemma 2.2 ([14, Lemma 5]) *Suppose that μ is a finite Borel measure on $[0,1]^2$, and that E is a subset of $(I\times J)^{\mathbf{N}}$ such that $K_T(E)$ is a Borel subset of $[0,1]^2$, and $\mu(K_T(E))>0$. Let δ be a positive number. For each point $x\in E$, put*

$$A(x)=\liminf_{k\to\infty}(k\delta+\log_m\mu(Q_k(x))).$$

1) *If $A(x)=-\infty$ for all $x\in E$, then $\mathcal{P}^\delta(K_T(E))=+\infty$;*
2) *If $A(x)=+\infty$ for all $x\in E$, then $\mathcal{P}^\delta(K_T(E))=0$;*
3) *If there are real numbers a and b such that $a\le A(x)\le b$ for all $x\in E$, then $0<\mathcal{P}^\delta(K_T(E))<+\infty$.*

The Borel measures on $[0,1]^2$ to which the above lemmas will be applied are constructed as follows. For a probability vector $\mathbf{p}=(p_d)_{d\in D}$ on D, let $\mu_{\mathbf{p}}$ be the infinite product probability measure on $D^{\mathbf{N}}$ determined by $\mathbf{p}$. Let $\widetilde{\mu}_{\mathbf{p}}$ be the Borel probability measure induced by the map K_T, i.e., $\widetilde{\mu}_{\mathbf{p}}(A)=\mu_{\mathbf{p}}\big(K_T^{-1}(A)\big)$ for any Borel set $A\subseteq K(T,D)$. One may, of course, regard $\widetilde{\mu}_{\mathbf{p}}$ as a Borel measure on $[0,1]^2$.

By means of the Law of Large Numbers, we have

$$\widetilde{\mu}_{\mathbf{p}}(K_T(\Omega(\Sigma))=1\quad\text{if}\quad\mathbf{p}\in\Sigma,$$

and

$$\widetilde{\mu}_{\mathbf{p}}(K_T(\Omega_s(\Sigma))=1\quad\text{if}\quad\mathbf{p}\in\Sigma^*.$$

In addition, when $\mathbf{p}=(p_d)_{d\in D}>0$ from the definition of $\widetilde{\mu}_{\mathbf{p}}$ it follows that for any $x=(x_i)_{i=1}^\infty\in D^{\mathbf{N}}$ (cf. formula (1.4) in [14], also formula (4.4) in [4])

$$\widetilde{\mu}_{\mathbf{p}}(Q_k(x))=\prod_{j=1}^{[\alpha k]}p_{x_j}\cdot\prod_{j=[\alpha k]+1}^{k}q_{\sigma(x_j)}.\tag{2.1}$$

Lemma 2.3 *Let Γ be a closed convex subset of H. Then there exists a unique $\mathbf{p}^*=(p_d^*)_{d\in D}\in\Gamma$ such that $g(\mathbf{p}^*)=g_{\max}(\Gamma)$ and*

$$\sum_{d\in D}p_d\left(-\alpha\log_m p_d^*-(1-\alpha)\log_m q^*_{\sigma(d)}\right)\le\sum_{d\in D}p_d^*\left(-\alpha\log_m p_d^*-(1-\alpha)\log_m q^*_{\sigma(d)}\right)\tag{2.2}$$

for all $\mathbf{p}=(p_d)_{d\in D}\in\Gamma$. In addition, if there exists a $\mathbf{p}\in\Gamma$ such that $\mathbf{p}>0$, then $\mathbf{p}^*>0$.

Proof. The existence and uniqueness of $\mathbf{p}^*$ satisfying $g(\mathbf{p}^*)=g_{\max}(\Gamma)$ are easily obtained by the compactness of Γ and strict concavity of $g(\mathbf{p})$.

Suppose that $\mathbf{p}=(p_d)_{d\in D}\in\Gamma$ such that $\mathbf{p}>0$. We claim that $D_1=\{d\in D:p_d^*=0\}$ is empty. Otherwise, both D_1 and $D_2=D\setminus D_1$ are nonempty. Let $\mathbf{p}_t=t\mathbf{p}+(1-t)\mathbf{p}^*=(tp_d+(1-t)p_d^*)_{d\in D},\ t\in[0,1]$. Then $\mathbf{p}_t\in\Gamma$ with $\mathbf{p}_t>0$ for $t\in(0,1]$ and $\mathbf{p}_0=\mathbf{p}^*$. Let $f(t)=g(\mathbf{p}_t)$, $t\in[0,1]$. Then one can check that $\lim_{t\to0+}f'(t)=+\infty$ which implies that $g(\mathbf{p}^*)$ cannot attain its maximum at $\mathbf{p}^*$.

Finally we prove (2.2). For any given $\mathbf{p} \in \Gamma$ let

$$y(t) = g(\mathbf{p}^* + t(\mathbf{p} - \mathbf{p}^*)), \quad 0 \le t \le 1.$$

Then $y(t)$ attains maximum at $t = 0$. So

$$\begin{aligned} 0 \ge y'(0) &= \langle g'(\mathbf{p}^*), \mathbf{p} - \mathbf{p}^* \rangle \\ &= \left\langle \left(-\alpha \log_m p_d^* - (1-\alpha) \log_m q_{\sigma(d)}^* - \frac{1}{\log m} \right)_{d \in D}, (p_d - p_d^*)_{d \in D} \right\rangle \end{aligned}$$

yielding (2.2). □

We like to point out that in [14] the proof of [R1] of Theorem A was only given for the case $\mathbf{p} > 0$. For completeness, we supplement the proof for the case $\mathbf{p} = (p_d)_{d \in D} \in H$ with some $p_d = 0$.

Lemma 2.4 *Let $\mathbf{p} = (p_d)_{d \in D} \in H$ with some $p_d = 0$. Then*

$$\dim_H K_T(\Omega(\{\mathbf{p}\})) = \dim_P K_T(\Omega(\{\mathbf{p}\})) = g(\mathbf{p}).$$

Proof. Let $D_1 = \{d \in D : p_d = 0\}$ and $D_2 = D \setminus D_1$. Let $\widetilde{\mathbf{p}} = (p_d)_{d \in D_2}$. Let $\widetilde{\Omega}(\widetilde{\mathbf{p}})$ be given by (1.5) with D_2 instead of D. Then $\dim_P K_T(\Omega(\{\mathbf{p}\})) \ge \dim_H K_T(\Omega(\{\mathbf{p}\})) \ge \dim_H K_T(\widetilde{\Omega}(\widetilde{\mathbf{p}})) = g(\mathbf{p})$.

In the following we show that $\dim_P K_T(\Omega(\{\mathbf{p}\})) \le g(\mathbf{p})$. Fix an $\epsilon > 0$. Take $\overline{\mathbf{p}} = (\bar{p}_d)_{d \in D} \in H$ such that $\overline{\mathbf{p}} > 0$ and

$$g_{\max}([\mathbf{p}, \overline{\mathbf{p}}]) \le g(\mathbf{p}) + \epsilon,$$

where $[\mathbf{p}, \overline{\mathbf{p}}] = \{(t\bar{p}_d + (1-t)p_d)_{d \in D} : 0 \le t \le 1\}$. Let $\mathbf{p}^* = (p_d^*)_{d \in D} \in [\mathbf{p}, \overline{\mathbf{p}}]$ be such that $g(\mathbf{p}^*) = g_{\max}([\mathbf{p}, \overline{\mathbf{p}}])$. Then $\mathbf{p}^* > 0$ by Lemma 2.3. To finish the proof one only needs to show that $\dim_P K_T(\Omega([\mathbf{p}, \overline{\mathbf{p}}])) \le g(\mathbf{p}^*)$. Note that for any $x = (x_i)_{i=1}^{\infty} \in \Omega([\mathbf{p}, \overline{\mathbf{p}}])$

$$\begin{aligned} \log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) &= \sum_{i=1}^{[\alpha k]} \log_m p_{x_i}^* + \sum_{i=[\alpha k]+1}^{k} \log_m q_{\sigma(x_i)}^* \\ &= \sum_{d \in D} N_{[\alpha k]}(x, d) \log_m p_d^* + \sum_{d \in D} N_k(x, d) \log_m q_{\sigma(d)}^* - \sum_{d \in D} N_{[\alpha k]}(x, d) \log_m q_{\sigma(d)}^*. \end{aligned}$$

Therefore, for each $x \in \Omega([\mathbf{p}, \overline{\mathbf{p}}])$

$$\begin{aligned} \lim_{k \to \infty} \frac{1}{k} \log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) &= \alpha \sum_{d \in D} f(x, d) \log_m p_d^* + (1-\alpha) \sum_{d \in D} f(x, d) \log_m q_{\sigma(d)}^* \\ &\ge \sum_{d \in D} p_d^* \left(\alpha \log_m p_d^* + (1-\alpha) \log_m q_{\sigma(d)}^* \right) = -g(\mathbf{p}^*). \end{aligned}$$

It follows from Lemma 2.2 that

$$\dim_P K_T(\Omega([\mathbf{p}, \overline{\mathbf{p}}])) \le g(\mathbf{p}^*).$$

So the proof is completed. □

Note that in Theorem B one requires $\mathbf{c} > 0$. The following lemma shows that it is still true when some c_i are zeros.

Lemma 2.5 *Let $\mathbf{c} = (c_i)_{i=1}^{s} \in H_s$ with some $c_i = 0$. Let $\mathbf{p} = (p_d)_{d \in D}$ be determined by (1.9) and let $(q_b)_{b \in B}$ be the probability vector induced by $(p_d)_{d \in D}$ via (1.3). Then*

$$\dim_H K_T(\Omega_s(\{\mathbf{c}\})) = g_{\max}(\{\mathbf{c}\}^*) = g(\mathbf{p}) = \alpha \sum_{j=1}^{s} \left(c_j \log_m \sum_{d \in \Gamma_j} q_{\sigma(d)}^{\theta} - c_j \log_m c_j \right),$$

where $\theta = \frac{\alpha - 1}{\alpha} = 1 - \log_m n$.

Proof. Let $D_1 = \{1 \le i \le s : c_i = 0\}$ and $D_2 = \{1, \dots, s\} \setminus D_1$. Let $\widetilde{\mathbf{c}} = (c_i)_{i\in D_2}$. Let $\Omega_{|D_2|}(\{\widetilde{\mathbf{c}}\})$ be given by (1.8) with $\bigcup_{i\in D_2}\Gamma_i$ instead of D and the corresponding partition Γ_i, $i \in D_2$. Then

$$\dim_H K_T(\Omega_s(\{\mathbf{c}\})) \ge \dim_H K_T(\Omega_{|D_2|}\{\widetilde{\mathbf{c}}\}) = g_{\max}(\{\mathbf{c}\}^*).$$

In the following we show that $\dim_H K_T(\Omega_s(\{\mathbf{c}\})) \le g_{\max}(\{\mathbf{c}\}^*)$. Fix an $\epsilon > 0$. Take $\bar{\mathbf{c}} = (\bar{c}_i)_{i=1}^s \in H_s$ such that $\bar{\mathbf{c}} > 0$ and

$$g_{\max}([\mathbf{c}, \bar{\mathbf{c}}]^*) \le g_{\max}(\{\mathbf{c}\}^*) + \epsilon,$$

where $[\mathbf{c}, \bar{\mathbf{c}}] = \{(t\bar{c}_i + (1-t)c_i)_{i=1}^s : 0 \le t \le 1\}$.

Let $\overline{\mathbf{p}} = (\bar{p}_d)_{d\in D} \in [\mathbf{c}, \bar{\mathbf{c}}]^*$ be such that $g(\overline{\mathbf{p}}) = g_{\max}([\mathbf{c}, \bar{\mathbf{c}}]^*)$. Then $\overline{\mathbf{p}} > 0$ by Lemma 2.3 and satisfies

$$\bar{p}_d = \frac{\bar{q}^{\theta}_{\sigma(d)}}{\sum_{d'\in\Gamma_j} \bar{q}^{\theta}_{\sigma(d')}} a_j, \quad \text{for } d \in \Gamma_j \text{ and } j = 1, 2, \dots, s. \tag{2.3}$$

Note that $a_i = \sum_{d\in\Gamma_i} \bar{p}_d,\ i = 1, \dots, s$. To finish the proof one only needs to show that $\dim_H K_T(\Omega_s([\mathbf{c}, \bar{\mathbf{c}}]^*)) \le g(\overline{\mathbf{p}})$. By (2.3) we have

$$\log_m \bar{p}_d = \theta \log_m \bar{q}_{\sigma(d)} + \log_m a_j - \log_m \sum_{d\in\Gamma_j} \bar{q}^{\theta}_{\sigma(d)}, \text{ for } d \in \Gamma_j,\ j = 1, 2, \dots, s. \tag{2.4}$$

By the definition of $N_k(x, \Gamma_j)$ in (1.1), we have that for any $x = (x_i)_{i=1}^{\infty} \in D^{\mathbf{N}}$ and any $k \in \mathbf{N}$

$$N_k(x, \Gamma_j) = \sum_{d\in\Gamma_j} N_k(x, d), \quad j = 1, 2, \dots, s.$$

Let

$$S_j(k) = \sum_{d\in\Gamma_j} N_k(x, d) \log_m \bar{q}_{\sigma(d)}, \quad j = 1, 2, \dots, s. \tag{2.5}$$

For $x = (x_i)_{i=1}^{\infty} \in \Omega_s([\mathbf{c}, \bar{\mathbf{c}}]^*)$, we have

$$\begin{aligned}
\log_m \widetilde{\mu}_{\overline{\mathbf{p}}}(Q_k(x)) &= \sum_{i=1}^{[\alpha k]} \log_m \bar{p}_{x_i} + \sum_{i=[\alpha k]+1}^{k} \log_m \bar{q}_{\sigma(x_i)} \\
&= \sum_{j=1}^{s}\sum_{d\in\Gamma_j} N_{[\alpha k]}(x, d) \log_m \bar{p}_d + \sum_{j=1}^{s}\sum_{d\in\Gamma_j} N_k(x, d) \log_m \bar{q}_{\sigma(d)} \\
&\quad - \sum_{j=1}^{s}\sum_{d\in\Gamma_j} N_{[\alpha k]}(x, d) \log_m \bar{q}_{\sigma(d)} \\
&= \sum_{j=1}^{s}\sum_{d\in\Gamma_j} N_{[\alpha k]}(x, d) \log_m a_j - \sum_{j=1}^{s} \log_m \sum_{d\in\Gamma_j} \bar{q}^{\theta}_{\sigma(d)} \sum_{d\in\Gamma_j} N_{[\alpha k]}(x, d) \\
&\quad + \sum_{j=1}^{s}\sum_{d\in\Gamma_j} N_k(x, d) \log_m \bar{q}_{\sigma(d)} - \sum_{j=1}^{s}\sum_{d\in\Gamma_j} \frac{N_{[\alpha k]}(x, d)}{\alpha} \log_m \bar{q}_{\sigma(d)} \\
&= \sum_{j=1}^{s} N_{[\alpha k]}(x, \Gamma_j) \log_m a_j - \sum_{j=1}^{s} N_{[\alpha k]}(x, \Gamma_j) \log_m \sum_{d\in\Gamma_j} \bar{q}^{\theta}_{\sigma(d)} \\
&\quad + \sum_{j=1}^{s} S_j(k) - \sum_{j=1}^{s} \frac{1}{\alpha} S_j([\alpha k]),
\end{aligned}$$

by (2.1), (2.4) and (2.5). Note that for each $x \in \Omega_s([\mathbf{c}, \bar{\mathbf{c}}]^*)$, we have

$$\limsup_{k\to\infty} \frac{1}{k} \log_m \widetilde{\mu}_{\overline{\mathbf{p}}}(Q_k(x)) = \sum_{j=1}^{s} \alpha f(x, \Gamma_j) \log_m a_j - \sum_{j=1}^{s} \alpha f(x, \Gamma_j) \log_m \sum_{d\in\Gamma_j} \bar{q}^{\theta}_{\sigma(d)} + \limsup_{k\to\infty} \sum_{j=1}^{s} \left(\frac{S_j(k)}{k} - \frac{S_j([\alpha k])}{\alpha k} \right). \tag{2.6}$$

Claim. For any $(z_i)_{i=1}^s \in [\mathbf{c}, \bar{\mathbf{c}}]$

$$\sum_{j=1}^{s} \alpha z_j \log_m a_j - \sum_{j=1}^{s} \alpha z_j \log_m \sum_{d\in\Gamma_j} \bar{q}^{\theta}_{\sigma(d)} \geq \sum_{j=1}^{s} \alpha a_j \log_m a_j - \sum_{j=1}^{s} \alpha a_j \log_m \sum_{d\in\Gamma_j} \bar{q}^{\theta}_{\sigma(d)}.$$

□

Proof of the claim. Take $\Gamma = [\mathbf{c}, \bar{\mathbf{c}}]^*$ and take $\mathbf{p} = (p_d)_{d\in D} \in \Gamma$ such that $\sum_{d\in\Gamma_j} p_d = z_j$. Then by Lemma 2.3 we have

$$\sum_{d\in D} p_d \left(\alpha \log_m \bar{p}_d + (1-\alpha) \log_m \bar{q}_{\sigma(d)}\right) \geq \sum_{d\in D} \bar{p}_d \left(\alpha \log_m \bar{p}_d + (1-\alpha) \log_m \bar{q}_{\sigma(d)}\right) = -g(\overline{\mathbf{p}}).$$

By putting $\bar{p}_d = \frac{\bar{q}^{\theta}_{\sigma(d)}}{\sum_{d'\in\Gamma_j} \bar{q}^{\theta}_{\sigma(d')}} a_j$ (see (2.3)) we have

$$\sum_{d\in D} p_d \left(\alpha \log_m \bar{p}_d + (1-\alpha) \log_m \bar{q}_{\sigma(d)}\right) = \sum_{j=1}^{s} \alpha z_j \log_m a_j - \sum_{j=1}^{s} \alpha z_j \log_m \sum_{d\in\Gamma_j} \bar{q}^{\theta}_{\sigma(d)}$$

and

$$\sum_{d\in D} \bar{p}_d \left(\alpha \log_m \bar{p}_d + (1-\alpha) \log_m \bar{q}_{\sigma(d)}\right) = \sum_{j=1}^{s} \alpha a_j \log_m a_j - \sum_{j=1}^{s} \alpha a_j \log_m \sum_{d\in\Gamma_j} \bar{q}^{\theta}_{\sigma(d)}.$$

So the claim is proved.

For a fixed $x \in \Omega_s([\mathbf{c}, \bar{\mathbf{c}}]^*)$, $\limsup_{k\to\infty} \sum_{j=1}^{s} \frac{S_j(k)}{k}$ is finite. Thus

$$\limsup_{k\to\infty} \sum_{j=1}^{s} \left(\frac{S_j(k)}{k} - \frac{S_j([\alpha k])}{\alpha k} \right) \geq \limsup_{k\to\infty} \sum_{j=1}^{s} \frac{S_j(k)}{k} - \limsup_{k\to\infty} \sum_{j=1}^{s} \frac{S_j([\alpha k])}{\alpha k} = 0.$$

Thus, by (2.6) and the above claim we have

$$\limsup_{k\to\infty} \frac{1}{k} \log_m \widetilde{\mu}_{\overline{\mathbf{p}}}(Q_k(x)) \geq -g(\overline{\mathbf{p}})$$

It follows from Lemma 2.1 that $\dim_H K_T(\Omega_s([\mathbf{c}, \bar{\mathbf{c}}]^*)) \leq g(\overline{\mathbf{p}})$. □

Finally, we consider a special partition $\{\Gamma_i\}_{i=1}^s$ of D. We take $\{\Gamma_i\}_{i=1}^s$ as the horizontal fibers of D, i.e., each Γ_i is a horizontal fiber of D. In this case, by (2.5) we have $S_j(k) = N_k(x, \Gamma_j) \log_m \bar{q}_{\sigma(d)}$, $j = 1, 2, \ldots, s (d \in \Gamma_j)$. Therefore, $\lim_{k\to\infty} \frac{1}{k} \log_m \widetilde{\mu}_{\overline{\mathbf{p}}}(Q_k(x)) = -g(\overline{\mathbf{p}})$. Thus one actually obtains

Corollary 2.6 *Let $\{\Gamma_i\}_{i=1}^s$ be the horizontal fibers of D. For $\mathbf{c} = (c_i)_{i=1}^s \in H_s$*

$$\dim_H K_T(\Omega_s(\{\mathbf{c}\})) = \dim_P K_T(\Omega_s(\{\mathbf{c}\})) = \sum_{j=1}^{s} \left(c_j \log_m |\Gamma_j|^{\alpha} - c_j \log_m c_j\right),$$

where $\alpha = \log_n m$.

3 Proofs

This section is mainly devoted to determining the Hausdorff dimensions of $K_T(\Omega(\Sigma))$ and $K_T(\Omega_s(\Sigma))$.

Theorem 3.1 *Let Σ be a nonempty subset of H. Then*

$$\dim_H K_T(\Omega(\Sigma)) = \dim_P K_T(\Omega(\Sigma)) = \sup_{\mathbf{p}\in\Sigma} \dim_H K_T(\Omega(\{\mathbf{p}\})) = g_{\sup}(\Sigma).$$

P r o o f. We first show that the result is true if Σ is closed and convex and there exists a $\mathbf{p} \in \Sigma$ with $\mathbf{p} > 0$. Let $\mathbf{p}^* = (p_d^*)_{d\in D} \in \Sigma$ be such that $g(\mathbf{p}^*) = \max_{\mathbf{p}\in\Sigma} g(\mathbf{p})$. Then $\mathbf{p}^* > 0$ by Lemma 2.3. One only needs to show that $\dim_P K_T(\Omega(\Sigma)) \le g(\mathbf{p}^*)$. Note that for $x = (x_i)_{i=1}^\infty \in \Omega(\Sigma)$

$$\begin{aligned}\log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) &= \sum_{i=1}^{[\alpha k]} \log_m p_{x_i}^* + \sum_{i=[\alpha k]+1}^{k} \log_m q_{\sigma(x_i)}^* \\ &= \sum_{d\in D} N_{[\alpha k]}(x,d)\log_m p_d^* + \sum_{d\in D} N_k(x,d)\log_m q_{\sigma(d)}^* \\ &\quad - \sum_{d\in D} N_{[\alpha k]}(x,d)\log_m q_{\sigma(d)}^*.\end{aligned}$$

Therefore, for each $x \in \Omega(\Sigma)$

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{k}\log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) &= \alpha \sum_{d\in D} f(x,d)\log_m p_d^* + (1-\alpha)\sum_{d\in D} f(x,d)\log_m q_{\sigma(d)}^* \\ &\ge \sum_{d\in D} p_d^* \left(\alpha \log_m p_d^* + (1-\alpha)\log_m q_{\sigma(d)}^*\right) = -g_{\max}(\Sigma).\end{aligned}$$

It follows from Lemma 2.2 that

$$\dim_P K_T(\Omega(\Sigma)) \le g_{\max}(\Sigma).$$

For the general case, let $\overline{\Sigma}$ be the closure of Σ. We will prove that $\dim_P K_T(\Omega(\Sigma)) \le g_{\max}(\overline{\Sigma})$ since $g_{\sup}(\Sigma) = g_{\max}(\overline{\Sigma})$. Note that $g(\mathbf{p})$ is uniformly continuous on H. For any $\epsilon > 0$ there exists a $\delta > 0$ such that $|g(\mathbf{p_1}) - g(\mathbf{p_2})| < \epsilon$ for any $\mathbf{p_1}, \mathbf{p_2} \in H$ with $|\mathbf{p_1} - \mathbf{p_2}| \le 2\delta$.

Let $B_\delta(\mathbf{p})$ be the closed ball of radius δ and centered at $\mathbf{p}$. Take a finite members, denoted by Σ_i, $1 \le i \le \ell$, from $\{B_\delta(\mathbf{p}) \cap H : \mathbf{p} \in \overline{\Sigma}\}$ such that $\bigcup_{i=1}^{\ell} \Sigma_i \supseteq \overline{\Sigma}$. Then

(I) each $\Sigma_i \subseteq H$ is compact and convex and there exists a $\mathbf{p} \in \Sigma_i$ with $\mathbf{p} > 0$;

(II) $g_{\max}(\Sigma_i) \le g_{\max}(\Sigma_i \cap \overline{\Sigma}) + \epsilon \le g_{\max}(\overline{\Sigma}) + \epsilon = g_{\sup}(\Sigma) + \epsilon$.

Let $\mathbf{p}_i \in \Sigma_i$ be such that $g(\mathbf{p}_i) = g_{\max}(\Sigma_i)$. Then the previous argument shows that

$$\dim_P K_T(\Omega(\Sigma_i)) \le g_{\max}(\Sigma_i) \le g_{\sup}(\Sigma) + \epsilon, \quad 1 \le i \le \ell,$$

leading to $\dim_P K_T(\Omega(\Sigma)) \le \max_{1\le i\le\ell} \dim_P K_T(\Omega(\Sigma_i)) \le g_{\sup}(\Sigma) + \epsilon$. □

Theorem 3.2 *Denote $\gamma = \dim_H K_T(\Omega(\Sigma)) = \dim_P K_T(\Omega(\Sigma))$. Let $\Sigma_{\sup}$ be defined as in* (1.6).

(I) *Suppose that $\Sigma_{\sup} \cap \Sigma \ne \emptyset$. We have*

(Ia) *If $(1/|D|, \dots, 1/|D|) \in \Sigma_{\sup} \cap \Sigma$ and if D has uniform horizontal fibers, then $0 < \mathcal{H}^\gamma(K_T(\Omega(\Sigma))) \le \mathcal{P}^\gamma(K_T(\Omega(\Sigma))) < \infty$;*

(Ib) *If $(1/|D|, \dots, 1/|D|) \in \Sigma_{\sup} \cap \Sigma$ but D has not uniform horizontal fibers, then $\mathcal{H}^\gamma(K_T(\Omega(\Sigma))) = \mathcal{P}^\gamma(K_T(\Omega(\Sigma))) = \infty$;*

(Ic) *If $(1/|D|, \dots, 1/|D|) \notin \Sigma_{\sup} \cap \Sigma$, then $\mathcal{H}^\gamma(K_T(\Omega(\Sigma))) = \mathcal{P}^\gamma(K_T(\Omega(\Sigma))) = \infty$;*

(II) *If $\Sigma_{\sup} \cap \Sigma = \emptyset$, then $\mathcal{H}^\gamma(K_T(\Omega(\Sigma))) = \mathcal{P}^\gamma(K_T(\Omega(\Sigma))) = 0$.*

Proof. (I) When $(1/|D|, \dots, 1/|D|) \in \Sigma_{\sup} \cap \Sigma$ and D has uniform horizontal fibers, one can check that

$$\gamma = g\left((1/|D|, \dots, 1/|D|)\right) = \alpha \log_m |D| + (1-\alpha) \log_m |B|.$$

Set $\mathbf{p} = (1/|D|, \dots, 1/|D|)$. Then for every $x \in \Omega(\Sigma)$ and $k \in \mathbf{N}$

$$\begin{aligned} k\gamma + \log_m \widetilde{\mu}_{\mathbf{p}}(Q_k(x)) &= k(\alpha \log_m |D| + (1-\alpha) \log_m |B|) \\ &\quad + [\alpha k] \log_m \frac{1}{|D|} + (k - [\alpha k]) \log_m \frac{1}{|B|} \\ &= (\alpha k - [\alpha k]) \log_m \frac{|D|}{|B|}, \end{aligned}$$

leading to

$$0 \le \liminf_{k\to\infty} \{k\gamma + \log_m \widetilde{\mu}_p(Q_k(x))\} \le \limsup_{k\to\infty} \{k\gamma + \log_m \widetilde{\mu}_p(Q_k(x))\} \le \log_m \frac{|D|}{|B|}.$$

Thus (Ia) follows from Lemmas 2.1 and 2.2.

For the cases (Ib) and (Ic), take $\mathbf{p} \in \Sigma_{\sup} \cap \Sigma$. Then

$$\mathcal{P}^{\gamma}(K_T(\Omega(\Sigma))) \ge \mathcal{H}^{\gamma}(K_T(\Omega(\Sigma))) \ge \mathcal{H}^{\gamma}(K_T(\Omega(\{\mathbf{p}\}))) = \infty$$

by (b) of [R3].

(II) For $k \in \mathbf{N}$ let $B_k(\mathbf{p})$ be the closed ball of radius $1/k$ and centered at $\mathbf{p}$. Let $\Sigma_k = \Sigma \setminus \bigcup_{\mathbf{p}\in\Sigma_{\sup}} B_k(\mathbf{p})$. Note that $\Sigma_{\sup}$ is compact. Then

$$\Sigma = \bigcup_{k=1}^{\infty} \Sigma_k \quad \text{and} \quad g_{\sup}(\Sigma_k) < \gamma.$$

Thus, we have

$$\mathcal{H}^{\gamma}(K_T(\Omega(\Sigma))) = \lim_{k\to\infty} \mathcal{H}^{\gamma}(K_T(\Omega(\Sigma_k))) = 0,$$

and

$$\mathcal{P}^{\gamma}(K_T(\Omega(\Sigma))) = \lim_{k\to\infty} \mathcal{P}^{\gamma}(K_T(\Omega(\Sigma_k))) = 0.$$

This completes the proof. □

The following theorem gives the variational formula for $\dim_H K_T(\Omega_s(\Sigma))$.

Theorem 3.3 *Let Σ be a nonempty subset of H_s. Let Σ^* and $\Omega_s(\Sigma)$ be defined as in* (1.7) *and* (1.8)*, respectively. Then*

$$\dim_H K_T(\Omega_s(\Sigma)) = g_{\sup}(\Sigma^*).$$

Proof. The inequality $\dim_H K_T(\Omega_s(\Sigma)) \ge g_{\sup}(\Sigma^*)$ is clear since for any $\mathbf{p} \in \Sigma^*$ we have

$$\dim_H K_T(\Omega_s(\Sigma)) \ge \dim_H K_T(\Omega(\{\mathbf{p}\})) = g(\mathbf{p})$$

by [R1] of Theorem A.

We first show that the result is true if Σ is closed and convex and there exists a $\mathbf{c} \in \Sigma$ with $\mathbf{c} > 0$. In this case, Σ^* is also closed and convex.

Let $\mathbf{p}^* = (p_d^*)_{d\in D} \in \Sigma^*$ be such that $g(\mathbf{p}^*) = g_{\max}(\Sigma^*)$. Then $\mathbf{p}^* > 0$ by Lemma 2.3. By the same argument as in the proof of Lemma 2.5 we have that for each $x = (x_k)_{k=1}^{\infty} \in \Omega_s(\Sigma)$

$$\limsup_{k\to\infty} \frac{1}{k} \log_m \widetilde{\mu}_{\mathbf{p}^*}(Q_k(x)) \ge -g(\mathbf{p}^*),$$

which implies that $\dim_H K_T(\Omega_s(\Sigma)) \le g(\mathbf{p}^*)$ by Lemma 2.1.

For the general case we will prove that $\dim_H K_T(\Omega_s(\Sigma)) \le g_{\max}(\overline{\Sigma^*})$ since $g_{\sup}(\Sigma^*) = g_{\max}(\overline{\Sigma^*})$.

Note that the function $g_{\max}(\{\mathbf{c}\}^*)$ in $\mathbf{c}$ is uniformly continuous on H_s. For any $\epsilon > 0$ there exists a $\delta > 0$ such that $|g_{\max}(\{\mathbf{c_1}\}^*) - g_{\max}(\{\mathbf{c_2}\}^*)| < \epsilon$ for any $\mathbf{c_1}, \mathbf{c_2} \in H_s$ with $|\mathbf{c_1} - \mathbf{c_2}| \leq 2\delta$.

Let $B_\delta(\mathbf{c})$ be the closed ball of radius δ and centered at $\mathbf{c} \in H_s$. Take a finite members, denoted by Σ_i, $1 \leq i \leq \ell$, from $\{B_\delta(\mathbf{c}) \cap H_s : \mathbf{c} \in \overline{\Sigma}\}$ such that $\bigcup_{i=1}^{\ell} \Sigma_i \supseteq \overline{\Sigma}$. Then

(I) each $\Sigma_i \subseteq H_s$ is compact and convex and there exists a $\mathbf{c} \in \Sigma_i$ with $\mathbf{c} > 0$;
(II) $g_{\max}(\Sigma_i^*) \leq g_{\max}((\Sigma_i \cap \overline{\Sigma})^*) + \epsilon \leq g_{\max}(\overline{\Sigma}^*) + \epsilon = g_{\sup}(\Sigma^*) + \epsilon$.

Then the previous argument shows that

$$\dim_H K_T(\Omega_s(\Sigma_i)) \leq g_{\max}(\Sigma_i^*) \leq g_{\sup}(\Sigma^*) + \epsilon, \quad 1 \leq i \leq \ell,$$

leading to $\dim_H K_T(\Omega_s(\Sigma)) \leq \max_{1\leq i\leq \ell} \dim_H K_T(\Omega_s(\Sigma_i)) \leq g_{\sup}(\Sigma^*) + \epsilon$. □

Acknowledgements The authors would like to thank the anonymous referees for their helpful comments and suggestions that lead to a significant improvement of the manuscript. The first author is supported by the excellent mid-youth innovative team project of Hubei Provincial Education Department #T201009. The second author is supported by the National Natural Science Foundation of China #10971069, #11271137 and Shanghai Education Committee Project 11ZZ41. The third author is supported by the National Natural Science Foundations of China (No. 11371248) and the RFDP of Higher Education of China grant (No. 20130073110074).

References

[1] K. Barański, Hausdorff dimension of the limit sets of some planar geometric constructions, Advances in Mathematics **210**, 215–245 (2007).
[2] T. Bedford, Crinkly Curves, Markov Partitions and Box Dimension in Self-similar Sets, Ph.D. Thesis, University of Warwick (1984).
[3] K. J. Falconer, The Hausdorff dimension of self-affine fractals, Math. Proc. Camb. Phil. Soc. **103**, 339–350 (1988).
[4] D. Gatzouras and S. P. Lalley, Hausdorff and box dimensions of certain self-affine fractals, Indiana University Math. J. **41**, 533–568 (1992).
[5] D. Gatzouras and S. P. Lalley, Statistically self-affine sets: Hausdorff and box dimensions, J. Theoret. Probab. **7**, 437–468 (1994).
[6] Y. X. Gui and W. X. Li, Hausdorff dimension of subsets with proportional fibre frequencies of the general Sierpinski carpet, Nonlinearity **20**, 2353–2364 (2007).
[7] Y. X. Gui and W. X. Li, A random version of McMullen-Bedford general Sierpinski carpets and its application, Nonlinearity **21**, 1745–1758 (2008).
[8] Y. X. Gui and W. X. Li, Hausdorff dimension of a class of subsets of the general Sierpinski carpet, Acta Sci. Math. (Szeged) **75**, 75–89 (2009).
[9] X. G. He, K. S. Lau, and H. Rao, Self-affine sets and graph-directed systems, Constr. Approx. **19**, 373–397 (2003).
[10] R. Kenyon and Y. Peres, Measures of full dimension on affine-invariant sets, Ergodic Theory Dyn. Systems **16**, 307–323 (1996).
[11] R. Kenyon and Y. Peres, Hausdorff dimensions of sofic affine-invariant sets, Israel J. Math. **122**, 540–574 (1996).
[12] J. King, The singular spectrum for general Sierpinski carpets, Advances in Mathematics **116**, 1–8 (1995).
[13] C. McMullen, The Hausdorff dimension of general Sierpinski carpets, Nagoya Math. J. **96**, 1–9 (1984).
[14] Ole A. Nielsen, The Hausdorff and packing dimensions of some sets related to Sierpinski carpets, Canad. J. Math. **51**, 1073–1088 (1999).
[15] L. Olsen, Self-affine multifractal Sierpinski sponges in $\mathbf{R}^d$, Pacific Journal of Mathematics **183**, 99–143 (1998).
[16] Y. Peres, The packing measure of self-affine carpets, Math. Proc. Camb. Phil. Soc. **115**, 437–450 (1994).
[17] Y. Peres, The self-affine carpets of McMullen and Bedford have infinite Hausdorff measure, Math. Proc. Camb. Phil. Soc. **116**, 513–526 (1994).